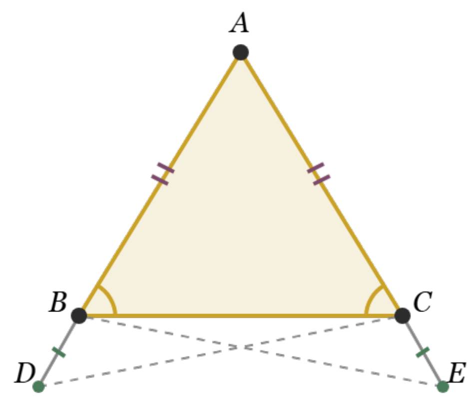
Lesson Plan: Proposition 5

In isosceles triangles the angles at the base equal one another; and if the equal sides are produced further, the angles under the base will equal one another.

Grade: Honors Geometry	Duration: 45-55 minutes	Type: Theorem



This is the famous 'Pons Asinorum' — the Bridge of Donkeys. For centuries, this was the proof where geometry students either crossed into real mathematical thinking or turned back. The claim seems obvious: if a triangle has two equal sides, the base angles must be equal. But proving it is genuinely tricky. Euclid's method is surprising — he doesn't use symmetry or folding. Instead, he extends the equal sides, constructs auxiliary segments, and applies SAS (Prop 4) twice. It's the first proof in Book I that requires real ingenuity.

Stage 1: Desired Results (UbD)

Essential Question

How can adding lines to a figure reveal hidden relationships that weren't visible before?

Learning Objectives

1. Prove that the base angles of an isosceles triangle are equal
2. Construct auxiliary lines (extending sides, cutting equal segments) to enable a proof
3. Apply Proposition 4 (SAS) to non-obvious triangle pairs within a figure
4. Explain the historical significance of the Pons Asinorum as a benchmark of geometric reasoning

Key Understandings

• Sometimes a proof requires building things that aren't in the original figure — auxiliary lines are a fundamental proof technique you'll use again and again

• A single diagram can contain multiple overlapping triangles, and spotting the right pair is a key proof skill

• The Pons Asinorum is a gateway: it's the first proof requiring genuine creativity, and it unlocks all subsequent work with isosceles and equilateral triangles

Vocabulary

Isosceles triangle, Base angles, Pons Asinorum (Bridge of Asses), Auxiliary construction, Vertex angle, SAS congruence (Prop 4)

Stage 2: Assessment Evidence

Formative Checks

• After the auxiliary construction: use colored pencils to outline the two large triangles and the two small triangles Euclid compares

• Mid-proof: write the SAS correspondence for each triangle pair before seeing the full argument

• Partner quiz: one person states a step, the other names the justification (Prop 3, Prop 4, CN2, or CN3)

Materials Needed

• Compass and straightedge for each student

• Pre-printed isosceles triangle diagrams (large, with room to extend sides)
• Colored pencils (at least three colors for marking different triangle pairs)
• Printed reference card summarizing Props 1-4

Stage 3: Learning Plan

Warm-Up (5 minutes)

Print a large isosceles triangle ABC with AB = AC. Try folding it along the line from A straight down to the base. The two halves match up and the base angles look equal. That fold is convincing — but it's not a proof. Today we prove this rigorously, and Euclid's method is more surprising than you'd expect. He doesn't fold. He extends.

Direct Instruction (15 minutes)

Here's how Euclid does it — use three colors, one for the original triangle, one for each SAS pair. Start with isosceles triangle ABC where AB = AC. (1) Extend AB beyond B to F, and AC beyond C to G. (2) Use Prop 3 to make BF = CG on the extensions. (3) Draw FC and GB. Now look at the two large triangles: AFC and AGB. Since AB = AC and BF = CG, then AF = AG (CN2). The angle at A is shared. AC = AB is given. By SAS (Prop 4), triangles AFC and AGB are congruent — so FC = GB and angle AFC = angle AGB. (4) Now the two small triangles: BFC and CGB. We have BF = CG (construction), FC = GB (just proved), and angle BFC = angle CGB (just proved). By SAS again, these are congruent — so angle FBC = angle GCB. (5) The punchline: angle ABC = angle ACB because they're supplements of equal angles. The base angles are equal.

Guided Practice (12 minutes)

Reconstruct the proof on the large printed diagrams. Working step by step: (1) extend the sides and mark equal segments using Prop 3, (2) draw the auxiliary segments FC and GB, (3) outline the first SAS pair in one color — write out the three equalities, (4) outline the second SAS pair in another color — write out those three equalities, (5) write the final conclusion. The hardest part is step 3 — identifying that first SAS pair. Take your time there.

Independent Practice (12 minutes)

Worksheet: (a) Given an isosceles triangle with specific measurements, write out the full Prop 5 proof with labels and justifications. (b) Why is this called the Bridge of Donkeys? What makes it harder than Props 1-3? (c) Identify the two applications of SAS and for each one, name the two sides and the included angle. (d) Challenge: modern textbooks prove this by drawing the altitude from the vertex and using SAS just once. Sketch that simpler proof. Why might Euclid have avoided it?

Closure (6 minutes)

Come back to the fold from the warm-up. The fold was correct — the base angles are equal. But Euclid proved it without folding, without assuming symmetry, without even assuming the altitude exists. He built auxiliary lines and applied SAS twice. That's the power of deductive reasoning — it works even when intuition can't guide you. New proof technique unlocked today: auxiliary constructions. You'll use this technique for the rest of Book I and beyond.

Differentiation: Supporting All Learners

For Struggling Students

Split the proof into two worksheets: Worksheet A covers the first SAS application (triangles AFC and AGB), Worksheet B covers the second (BFC and CGB). Work through A together, then attempt B with a partner. A color-coded diagram with each triangle pair pre-outlined makes the overlapping triangles much easier to see.

For Advanced Students

Write the modern proof using the angle bisector from A to the midpoint of BC. Compare it to Euclid's: which uses fewer propositions? Which is more elegant? Then try the converse: if the base angles are equal, must the triangle be isosceles? That's Proposition 6 — see if you can prove it using contradiction.

Tips: Teacher Notes

> **Teacher note:** This is the hardest proof so far. Budget time for confusion and re-explanation. It's normal for this lesson to run long.

> **Teacher note:** Three colors on the board is essential. Without color-coding, students genuinely cannot distinguish the overlapping triangles.

> **Teacher note:** Students will ask: 'Why not just draw the altitude and use symmetry?' Great question. Euclid hasn't proved that the altitude bisects the base yet — using that would be circular reasoning at this stage.

> **Teacher note:** The historical context resonates: for centuries this proof was the test of whether a student could handle real geometry. It validates the feeling that this is genuinely difficult — it's supposed to be.

> **Teacher note:** The challenge problem (the modern proof) makes excellent homework if time runs short.

Preview: Next Lesson

Next: Proposition 6 — If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

If a triangle has two equal angles, then the sides opposite those angles are also equal—making it isosceles. (This is the converse of Proposition 5.)