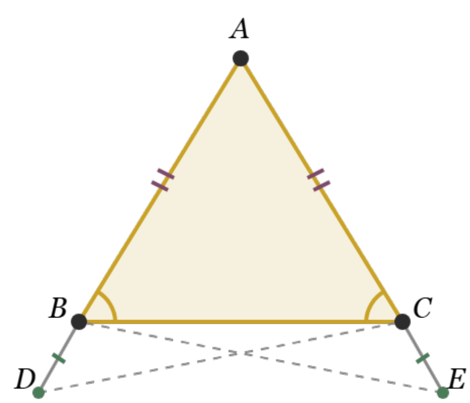
Answer Key: Proposition 5

In isosceles triangles the angles at the base equal one another; and if the equal sides are produced further, the angles under the base will equal one another.

Part A: Construction Practice — Notes

Solution Figure



Teacher note: This is the most complex proof in the first five propositions and marks a real step up in difficulty. Begin by drawing isosceles triangle ABC with AB = AC. Then extend AB beyond B to point D and AC beyond C to point E, constructing BD = CE using Prop 3. Students must track four triangles: the original ABC, the large triangles ADC and AEB, and the smaller triangles BDC and CEB. Color-coding each triangle pair helps enormously. The proof uses Prop 4 (SAS) twice, CN2 (addition), and CN3 (subtraction). Budget extra time for this proposition.

Part B: Proof Justification — Answers

Each step is matched with the correct toolkit justification.

Available Toolkit:

• D20 — Types of Triangles (by sides)

• P2 — Extend a Line

• Prop 3 — To cut off from the greater of two given unequal straight lines a straight line equal to the less.

• Prop 4 — If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively.

• CN1 — Transitivity of Equality

• CN2 — Addition Preserves Equality

• CN3 — Subtraction Preserves Equality

• P1 — Draw a Line

Step	Statement	Justification
1	Given: Triangle ABC with AB = AC (isosceles)	D20
2	Extend AB to D and AC to E.	P2
3	Make BD = CE.	Prop 3
4	Since AB = AC and BD = CE, then AB + BD = AC + CE.	CN2
5	Therefore AD = AE.	CN1
6	Join DC and EB.	P1
7	In triangles ADC and AEB: AD = AE, AC = AB, and DAC = EAB (the angle between the same two lines through A). Therefore DC = EB, ACD = ABE, and ADC = AEB.	Prop 4

8	In triangles BDC and CEB: BD = CE, DC = EB, and BDC = CEB (equal to ADC = AEB, since B lies on AD and C lies on AE). Therefore DBC = ECB and BCD = CBE.	Prop 4
9	Subtracting the equal angles CBE = BCD from the equal whole angles ABE = ACD gives ABC = ACB.	CN3

> **Teacher note:** The proof has eight steps organized in three phases. Phase 1 (setup): Given AB = AC (D20, isosceles definition), extend to D and E (P2) with BD = CE (Prop 3), then since AB = AC and BD = CE, AB + BD = AC + CE by CN2, so AD = AE by CN1. Phase 2 (large triangles): In triangles ADC and AEB, we have AD = AE, AC = AB, and the included angle at A is shared, so by Prop 4 (SAS), the triangles are congruent. This yields DC = EB and angle ACD = angle ABE. Phase 3 (small triangles and conclusion): In triangles BDC and CEB, we have BD = CE (constructed), DC = EB (from phase 2), and angle BDC = angle CEB (from phase 2), so by Prop 4 again, angle DBC = angle ECB. Finally, subtracting the equal exterior angles from the equal whole angles via CN3 gives angle ABC = angle ACB — the base angles are equal.

Why each justification works:

Step 1: D20 — Types of Triangles (by sides). What definition describes a triangle with two equal sides?

Step 2: P2 — Extend a Line. Which postulate lets us extend a straight line?

Step 3: Prop 3 — To cut off from the greater of two given unequal straight lines a straight line equal to the less.. Which proposition cuts off an equal segment?

Step 4: CN2 — Addition Preserves Equality. If equals are added to equals, the wholes are equal.

Step 5: CN1 — Transitivity of Equality. Things which equal the same thing also equal one another.

Step 6: P1 — Draw a Line. Which postulate lets us draw a straight line between two points?

Step 7: Prop 4 — If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively.. Which proposition proves triangles congruent by SAS?

Step 8: Prop 4 — If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively.. Same proposition — applied to the second pair of triangles.

Step 9: CN3 — Subtraction Preserves Equality. If equals are subtracted from equals, the remainders are equal.

Part C: Thinking Deeper — Model Answers

Question 1: Why is this called the 'Bridge of Asses'? What makes it a turning point?

> **Teacher note:** The name "Pons Asinorum" (Bridge of Asses) has two traditional explanations. First, the diagram with its extended lines resembles a steep bridge that only a sure-footed traveler can cross. Second, and more commonly cited, this proposition historically marked the point where students who could not handle rigorous geometric reasoning would stumble and give up — the "asses" who could not cross the bridge to more advanced mathematics.

Question 2: Can you think of a visual or physical way to see that the base angles must be equal?

> **Teacher note:** Folding the triangle along the perpendicular from A to BC is intuitively compelling, but Euclid cannot use it. He has not yet proven that a perpendicular from A to BC exists, that it bisects BC, or that reflection preserves distances and angles. Each of these facts requires its own proof. Euclid's longer approach using extensions and auxiliary triangles avoids all unproven assumptions, relying only on results already established in Props 1-4.

Question 3: What would it mean if an isosceles triangle had UNEQUAL base angles?

> **Teacher note:** Prop 5 directly uses P2 (to extend the sides), Prop 3 (to cut off equal extensions BD = CE), and Prop 4 (SAS, applied twice to establish congruence of the large and small triangle pairs). It also uses CN2 (adding equals), CN1 (things equal to the same thing), and CN3 (subtracting equal angles to get the base angles). Indirectly, it depends on the entire chain: Prop 3 requires Prop 2, which requires Prop 1. This is the first proposition that truly exercises the full machinery built in Props 1-4.

Part D: Challenge Problem — Model Answer

Teacher note: Proposition 6 proves the converse: if a triangle has two equal base angles, then the sides opposite those angles are equal (the triangle is isosceles). Euclid proves this by contradiction — assume the sides are unequal, use Prop 3 to cut the longer side down to match the shorter, then apply Prop 4 (SAS) to show a smaller triangle equals a larger one, contradicting CN5 (the whole is greater than the part). Together, Props 5 and 6 establish a biconditional: a triangle has equal sides if and only if it has equal base angles. This "if and only if" relationship means the two properties are completely interchangeable — either one can serve as the definition of isosceles.

Notes: Common Student Mistakes

• Using CN1 where CN2 is needed in step 4. Adding equal segments BD and CE to equal sides AB and AC to get AD = AE is addition of equals — that is CN2, not CN1. CN1 is then used in step 5 to conclude AD = AE from the equal sums.

• Confusing CN3 with CN1 in the final step. The conclusion subtracts equal angles (angle DBC and angle ECB) from equal angles (angle ABE and angle ACD) to get equal remainders (angle ABC and angle ACB). Subtraction of equals is CN3.

• Losing track of which triangles are being compared. There are two SAS applications: first comparing the large triangles ADC and AEB, then the smaller triangles BDC and CEB. Students often muddle which sides and angles belong to which comparison.

• Wanting to "just fold the triangle" instead of following Euclid's proof. While folding is good geometric intuition, Euclid cannot use it — he has not proven that a midpoint of BC exists, that the fold line is perpendicular, or that folding preserves angles. His construction with extensions avoids all these unproven assumptions.