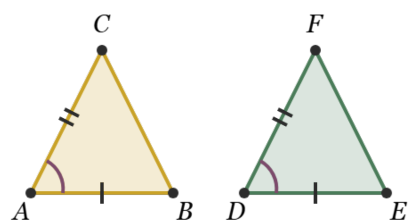
Lesson Plan: Proposition 4

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively.

Grade: 9-12	Duration: 45-55 minutes	Type: Theorem



This is where everything shifts. Props 1-3 were constructions — building things with compass and straightedge. Prop 4 is the first theorem: it proves something true about all triangles. The claim: if two triangles share two sides and the angle between them (Side-Angle-Side), they're identical in every way. Euclid proves it by imagining one triangle picked up and placed on the other — a method called 'superposition' that mathematicians have debated for centuries. Whether or not you find the proof perfectly rigorous, the result is rock-solid and gets used constantly from here on.

Stage 1: Desired Results (UbD)

Essential Question

What is the minimum information needed to know that two triangles are exactly the same shape and size?

Learning Objectives

1. State the SAS congruence criterion in precise language
2. Explain Euclid's superposition argument and why it differs from constructions
3. Apply SAS to determine whether two given triangles are congruent
4. Recognize that Prop 4 is a theorem, not a construction, and articulate the difference

Key Understandings

• Two sides and the angle between them uniquely determine a triangle — no other triangle can have those same three measurements without being congruent

• Superposition (placing one figure on top of another) is a proof method, though a controversial one — Euclid uses it only here and in Prop 8

• This is the first result in the Elements that proves something is true rather than constructing something new

Vocabulary

Congruent, Side-Angle-Side (SAS), Included angle, Superposition, Corresponding parts, Theorem vs. construction

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Stage 2: Assessment Evidence

Formative Checks

• Show five pairs of triangles with measurements. For each: does SAS apply, or is there not enough information? Why?

• Key question to discuss: 'Why does the angle have to be between the two sides? What goes wrong if it isn't?'

• Exit ticket: draw two triangles that satisfy SAS and mark the equal parts with tick marks and angle arcs

Materials Needed

• Printed pairs of triangles (some congruent by SAS, some not)

• Scissors and tracing paper for superposition demonstration
• Protractors and rulers (for measurement verification)
• Colored pencils for marking corresponding parts

Stage 3: Learning Plan

Warm-Up (7 minutes)

Here's an experiment: draw a triangle with two sides measuring 5 cm and 7 cm, meeting at a 40° angle. Use a protractor and ruler. Now have your student (or a partner) do the same thing independently — don't look at each other's work. When you're both done, cut out the triangles and stack them. They'll match perfectly. Ask: 'Why did we both get the same triangle? What information locked it in?' That question is exactly what Proposition 4 answers.

Direct Instruction (13 minutes)

Draw triangles ABC and DEF with AB = DE, AC = DF, and angle BAC = angle EDF. Here's Euclid's argument: imagine picking up triangle ABC and placing it on DEF so that A lands on D and side AB lies along DE. Since AB = DE, point B lands exactly on E. Since angle BAC = angle EDF, ray AC falls along ray DF. Since AC = DF, point C lands on F. All three vertices coincide, so the triangles match exactly — by Common Notion 4 (things that coincide are equal). Now address the elephant in the room: 'Is picking up a triangle and moving it really a proof? Mathematicians have argued about this for centuries. Euclid never states a postulate that says figures can be moved without changing. Modern treatments often just take SAS as an axiom instead of proving it.' It's worth being honest about this — it's one of the most interesting philosophical moments in Book I.

Guided Practice (12 minutes)

Work through three examples together. Each shows two triangles with some measurements. For each: (1) identify the two sides and the angle between them, (2) check whether SAS applies, (3) if yes, list all corresponding parts that must be equal. One example should be a non-example where the angle is NOT between the two equal sides — make sure to identify that SAS doesn't apply and talk about why. Tracing paper helps make the superposition idea concrete.

Independent Practice (12 minutes)

Worksheet: (a) Two applied problems — like a bridge truss with given measurements — where SAS determines whether parts are congruent. (b) A 'what's wrong with this proof?' exercise where someone incorrectly applies SAS with a non-included angle. (c) In your own words: Props 1-3 are constructions, Prop 4 is a theorem. What's the difference? (d) Mark corresponding congruent parts on three pairs of triangles.

Closure (6 minutes)

Come back to the warm-up: two sides and the included angle were enough to guarantee identical triangles. That's Prop 4. Add it to the toolkit chart: Prop 4 = SAS congruence. Then look ahead: 'If a triangle has two equal sides, what can we say about the angles opposite those sides? Think about it — that's Proposition 5, the famous Bridge of Donkeys.'

Differentiation: Supporting All Learners

For Struggling Students

Make a simple checklist: (1) Find two pairs of equal sides. (2) Is the equal angle between those sides? (3) If both yes, the triangles are congruent. Physical cut-outs help — literally placing one triangle on another makes superposition real instead of abstract.

For Advanced Students

The philosophical question: Euclid assumes we can move figures without changing them, but he never states this as a postulate. Is that a gap? Hilbert later replaced superposition with a formal axiom. Also: would SSA (side-side-angle, non-included) guarantee congruence? Try to find a counterexample — two triangles with the same SSA that aren't congruent.

Tips: Teacher Notes

Teacher note: This is a big conceptual shift — from 'build things' to 'prove things.' Name the shift explicitly so students feel the transition.

Teacher note: The warm-up is essential. Once students see empirically that two sides and the included angle determine a unique triangle, the theorem feels inevitable rather than arbitrary.

Teacher note: Students commonly confuse SAS with SSA. The non-example is worth extra time.

Teacher note: Update the toolkit chart: Prop 1 (equilateral triangle), Prop 2 (copy length), Prop 3 (trim to length), Prop 4 (SAS congruence).

Preview: Next Lesson

Next: Proposition 5 — In isosceles triangles the angles at the base equal one another; and if the equal sides are produced further, the angles under the base will equal one another.

If a triangle has two equal sides (isosceles), then the two angles opposite those equal sides are also equal. And if the equal sides are extended below the base, the angles under the base are equal too.