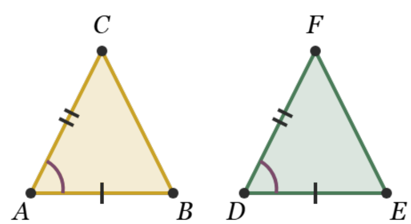
Answer Key: Proposition 4

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively.

Part A: Construction Practice — Notes

Solution Figure



Teacher note: This is the first theorem (as opposed to construction) in the Elements, so the classroom dynamic shifts from building to reasoning. No compass or straightedge is needed — instead, students reason about superposition. Set up with two triangles: ABC and DEF, where AB = DE, AC = DF, and angle BAC = angle EDF. Walk students through the mental operation of placing triangle ABC on DEF so that A lands on D and AB lies along DE. Use transparent overlays or cutout triangles to make superposition tangible. Emphasize that CN4 ("things which coincide with one another are equal") is the engine driving every step.

Part B: Proof Justification — Answers

Each step is matched with the correct toolkit justification.

Available Toolkit:

• CN4 — Coincidence Implies Equality

• D8 — Plane Angle

• P1 — Draw a Line

• D20 — Types of Triangles (by sides)

• CN1 — Transitivity of Equality

• P3 — Draw a Circle

Step	Statement	Justification
1	Place triangle ABC on triangle DEF, aligning A with D	CN4
2	Since AB = DE, point B coincides with E	CN4
3	Since BAC = EDF, ray AC falls along ray DF	D8
4	Since AC = DF, point C coincides with F	CN4
5	B coincides with E and C coincides with F, so BC coincides with EF	P1
6	The triangles coincide entirely, so all corresponding parts are equal	CN4

Teacher note: The proof proceeds by superposition in six steps. (1) Place triangle ABC on DEF with A on D, directing AB along DE. (2) Since AB = DE, point B lands exactly on E (P1 — a segment of given length determines its endpoint). (3) Since angle BAC = angle EDF (D8, definition of angle), ray AC falls along ray DF. (4) Since AC = DF, point C lands exactly on F. (5) Now B coincides with E and C coincides with F, so segment BC coincides with segment EF because two points determine a unique straight line (P1). (6) The triangles coincide completely, so they are equal in all respects (CN4). This establishes that all remaining sides and angles are equal — full congruence.

Why each justification works:

Step 1: CN4 — Coincidence Implies Equality. Placing one figure on another is Euclid's famous (tacit) superposition move — which common notion makes the method meaningful by equating coinciding figures?

Step 2: CN4 — Coincidence Implies Equality. Euclid tacitly uses the converse of this common notion: equal lengths, when placed together, coincide.

Step 3: D8 — Plane Angle. Which definition tells us about angles?

Step 4: CN4 — Coincidence Implies Equality. Same tacit converse as before: equal lengths, when placed together, coincide.

Step 5: P1 — Draw a Line. Euclid reads into this postulate that the line between two points is unique — two straight lines cannot enclose a space.

Step 6: CN4 — Coincidence Implies Equality. Things which coincide with one another are equal.

Part C: Thinking Deeper — Model Answers

Question 1: Why does Euclid need to 'move' one triangle onto another? Is this really a proof?

> **Teacher note:** The angle must be "included" — sandwiched between the two known sides. This is because the included angle determines where the two sides point relative to each other, which in turn fixes the position of the third vertex. If the angle were at the end of one side (not between them), the other side could swing to different positions, potentially creating two valid but non-congruent triangles.

Question 2: What if we knew two sides and a NON-included angle? Would that be enough?

> **Teacher note:** Two sides and a non-included angle (SSA) do not guarantee congruence. This is the "ambiguous case": depending on the lengths involved, there may be two distinct triangles satisfying the conditions, exactly one, or none at all. A physical demonstration helps — fix two sticks and a hinge at the wrong end, and the free end can reach two different positions on the opposite side.

Question 3: Why is the angle between the two sides (the 'included' angle) so important?

> **Teacher note:** Proposition 4 is one of the most-used results in Book I. It appears directly in Prop 5 (isosceles base angles), where two pairs of triangles are compared using SAS. It is also essential in Prop 8 (SSS congruence), Prop 16, Prop 26, and many others. It is the fundamental bridge between knowing some parts of two triangles are equal and concluding the rest are as well.

Part D: Challenge Problem — Model Answer

> **Teacher note:** The minimum information to completely determine a triangle is three independent measurements of the right type: SAS (two sides and the included angle), ASA (two angles and the included side), or SSS (all three sides). Each combination uniquely fixes the triangle's shape and size. SSA fails because it can be ambiguous, and AAA fails because it determines shape but not size (similar triangles, not congruent ones). Prop 4 establishes the SAS case; the others are proven in later propositions.

Notes: Common Student Mistakes

• Not recognizing CN4 as the central justification. CN4 (things that coincide are equal) appears multiple times — whenever two elements land on top of each other. Students often reach for other common notions instead.

• Writing D15 or D20 for step 3 (the angle alignment) instead of D8. D8 defines a plane angle; it is what allows us to say that equal angles direct rays along the same path. D15 is about circles, D20 about equilateral triangles — neither applies here.

• Misunderstanding the role of P1 in step 5. Here P1 is invoked not to draw a line, but to assert uniqueness: since B = E and C = F, there is exactly one line through these two points, so BC and EF must be the same line.

• Confusing SAS with SSA. If students try to use two sides and a non-included angle, point out that this configuration can yield two different triangles (the ambiguous case). The "included" angle — the one between the two known sides — is what locks the triangle into a unique shape.